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TRANSONIC FLOW THROUGH NEARLY CIRCULAR DUCTS WITH GRADUALLY VARYING CROSS-SECTION

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A METHOD FOR THE ANALYSIS OF A TRANSONIC FLOW FIELD IN A NEARLY CIRCULAR DUCT WITH GRADUAL CHANGES IN CROSS-SECTION IS DEVELOPED. A LINEARIZED DIFFERENTIAL EQUATION FOR THE DEVIATION FROM THE RESULTS OF THE ONE-DIMENSIONAL THEORY IS PRESENTED AND IS SOLVED BY THE SUPERPOSITION OF PARTICULAR SOLUTIONS OBTAINED BY A PRODUCT HYPOTHESIS. THE POTENTIAL EQUATION, IN THIS CASE, WAS SIMPLIFIED FOR THE VICINITY OF THE SONIC VELOCITY.

APPLICATION TO A CIRCULAR SUBSONIC-SUPERSONIC DUCT IS MADE IN ORDER TO DETERMINE THE MAGNITUDE OF PROPAGATED SUBSONIC DISTURBANCES AND THE RESULTING VELOCITY DISTRIBUTION.

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ABSTRACT

A method for the analysis of a transonic flow field in a nearly circular duct with gradual changes in cross-section is developed. A linearized differential equation for the deviation from the results of the one-dimensional theory is presented and is solved by the superposition of particular solutions obtained by a product hypothesis. The potential equation, in this case, was simplified for the vicinity of the sonic velocity.

Application to a circular subsonic-supersonic duct is made in order to determine the magnitude of propagated subsonic disturbances and the resulting velocity distribution.

PUBLICATION APPROVAL

For the Commanding General:

for Ernest M. Ljunggren Lt.Col. USAF
O. J. RITLAND
Colonel, USAF
Chief, Aircraft Laboratory
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FOREWORD

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TABLE OF CONTENTS

LIST OF SYMBOLS

a	velocity of sound
$a^*_{\nu\mu}$	functions of x fulfilling differential equation for φ
r	radius of the duct
x, y, ω	cylindrical coordinates
$Z_{\nu\mu}$	functions of y fulfilling differential equation for φ
γ	ratio of specific heats, for air $\gamma = 1.4$
Φ	total velocity potential
φ	perturbation velocity potential
φ_s	one-dimensional flow potential
$\bar{\varphi}$	deviation from one-dimensional flow potential

SUBSCRIPTS

x, y, ω	refer to partial derivatives with respect to the cylindrical coordinates
o	refers to symmetric duct configuration
ν	refers to unsymmetric duct configuration

SUPERSCRIPTS

*	refers to conditions at the duct throat
	refers to partial derivatives with respect to the x coordinate

SECTION I

INTRODUCTION

This report presents an analytical method for the calculation of the flow field in a nearly circular duct with relatively gradual changes in cross-section. Due to simplifications introduced by means of the transonic law of similarity, the method in its present form would be restricted to flow fields with Mach numbers close to unity.

The investigation was initiated in connection with a plan for modifying the 10-Foot Wind Tunnel at Wright-Patterson Air Force Base. It was intended to vary the cross-sectional area with small inserts, rather than by a variation of the radius. A method was evolved to obtain the magnitude of deviations from the ideal parallel flow which might be caused by such an insert design. Before the actual computations were made the design was modified. However, the method itself is of interest, in that there are many other applications. An application, referring mainly to a modified test section design, will be given in the second part of this report.

The present analysis begins with a basic flow as given by one-dimensional theory. Any deviations from the basic flow are considered small, so that they can be determined by a linearized partial differential equation. The general solution to the differential equation is expressed by the superposition of particular solutions, each of which is the product of a Bessel function dependent on the radial distance y from the duct axis, a trigonometric function of the angular position ω, and a function of the axial position, x. Of special interest is the last function which is influenced by the duct shape.

If one is interested only in supersonic flow analysis, an alternate method of computation is the well-known method of characteristics. The method of characteristics is more general, in that the condition of a slow change in cross-section is not required. The method of characteristics for three dimensions is similar to the present method in the requirement of a nearly circular cross-section. The major advantages of the present method are that the subsonic influences can be determined and that being essentially more analytical, a better picture of the flow behavior in general cases may be obtained. In some cases, depending upon the duct shape, the solutions can be obtained in a less tedious manner.

Briefly, the report starts with the simplification of the general flow differential equation by means of the transonic law of similarity. Then a general method of solution is developed, and finally this result is applied to some specific example.

SECTION II

THEORETICAL CONSIDERATIONS

Simplification of the Flow Differential Equation Near the Sonic Speed

This analysis is based on a cylindrical coordinate system (Figure 1). Let the $\bar{x}$ axis be the duct center line, $\bar{y}$, the radial distance from the $\bar{x}$ axis, and ω , the angular position measured clockwise from the vertical position. The velocity potential is denoted by Φ . Partial derivatives are denoted by subscripts (i.e., $\frac{\partial \Phi}{\partial \bar{x}} = \Phi_{\bar{x}}$)

The general three-dimensional equation for a steady potential flow (derived in Appendix A) is given by

$$(1.1)\quad \Phi_{\bar{x}\bar{x}}\left(1-\frac{\Phi_{\bar{x}}^2}{a^2}\right) - 2\,\Phi_{\bar{x}\bar{y}}\left(\frac{\Phi_{\bar{x}}\Phi_{\bar{y}}}{a^2}\right) + \Phi_{\bar{y}\bar{y}}\left(1-\frac{\Phi_{\bar{y}}^2}{a^2}\right) + \frac{\Phi_{\bar{y}}}{\bar{y}} + \frac{\Phi_{\omega\omega}}{\bar{y}^2} = 0$$

If deviations from a parallel sonic flow are small, the simplified transonic equation results. (Derivation may be found in Appendix B).

$$(1.2)\quad -(\gamma+1)\,\varphi_{xx}\,\varphi_x + \varphi_{yy} + \frac{\varphi_y}{y} + \frac{\varphi_{\omega\omega}}{y^2} = 0$$

where $\quad x = \frac{\bar{x}}{r_o}$

$y = \frac{\bar{y}}{r_o}$

$$(1.3)\quad \varphi(x,y) = \frac{1}{r_o a^*}\left(\Phi - a^*\bar{x}\right)$$

and r_o is the radius at the throat.

The Pressure and Density Relations

The new potential term $\varphi(x,y)$ is directly related to the velocity.

However, one may be interested in the pressure and density relations at various positions along the duct. Bernoulli's law for compressible non-viscous gases is given by

$$(1.4)\quad \frac{\Delta p}{\rho} + u\,\Delta u + v\,\Delta v = 0$$

The last term is negligible, based on earlier assumptions. Because $\Delta u = \varphi_x a^*$, one can write

$$\Delta p = -\rho^* a^* (\varphi_x a^*)$$

Since $\gamma \frac{p^*}{\rho^*} = a^*$, the preceding equation is rewritten as

(1.5) $\frac{\Delta p}{p^*} = -\gamma\, \varphi_x$

<u>The Boundary Conditions</u>

Let $\bar{r}$ be the value of y at a point on the duct surface so that $\bar{r} = \bar{r}\,(x, \omega)$. This function, in the case of a vertical line of symmetry is expressed as

(1.6) $\bar{r} = \bar{r}_0(x) + \sum_1^\infty \left(\bar{r}_n(x) \cos n\omega\right)$

The functions r_0, r_1, r_2, ... r_n are obtained by a Fourier analysis at the specific x positions, and are written as

(1.6a) $\bar{r}_0(x) = \frac{1}{2\pi} \int_0^{2\pi} \bar{r}\,(x, \omega)\, d\omega$

(1.6b) $\bar{r}_n(x) = \int_0^{2\pi} \bar{r}(x, \omega) \cos n\omega \; d\omega$

Let $r = \frac{\bar{r}}{r_0}$.

The boundary condition of zero flow normal to the duct surface must be fulfilled. If the differences from the sonic velocity are small, this condition is expressed as

(1.7) $-\frac{dr}{dx} + \varphi_y = 0$

(The derivation may be found in Appendix C).

Let us introduce for the slope

(1.8) $\theta = \frac{dr}{dx}$

Then one has the relation

(1.9) $\varphi_y = \theta_0 + \sum \theta_\nu \cos \nu\omega$

Where θ_ν is an additional slope for the unsymmetrical configurations.

<u>The Basic Hypothesis for the Solution</u>

The solution is based on one-dimensional flow theory, in which the velocity distribution over each cross section is considered constant. The cross section is written as functions of the distance from the

minimum cross-section. The potential of this basic flow is denoted by Φ_B, where $\frac{\partial \Phi_B}{\partial x} = \Phi_{B_x} = f(r, r_0)$. Specifically, as derived in Appendix D, this gives

$$(1.10) \quad \Phi_{B_x} = \left(\frac{4}{\gamma+1} \frac{\Delta r}{r_0}\right)^{\frac{1}{2}}$$

The entire potential is expressed by

$$(1.11) \quad \Phi = \Phi_B(x) + \bar{\Phi}(x, y, \omega)$$

The $\bar{\Phi}$ represents the deviations from the basic flow, and shall be considered small enough that higher order terms can be neglected. If the values of the complete potential (Equation (1.11)) is inserted into the transonic differential equation (Equation (1.2)) and second order terms of $\bar{\Phi}$ are neglected, the following inhomogeneous equation is given:

$$(1.12) \quad -(\gamma+1)(\Phi_{B_x} \bar{\Phi}_{xx} + \Phi_{B_{xx}} \bar{\Phi}_x) + \left(\bar{\Phi}_{yy} + \frac{\bar{\Phi}_y}{y} + \frac{\bar{\Phi}_{\omega\omega}}{y^2}\right) = (\gamma+1)\Phi_{B_x} \Phi_{B_{xx}}$$

For the representation of Φ, let us introduce the following functions. Let a Bessel function of ν th order and of the argument η be denoted by $J_\nu(\eta)$. Let $\bar{\eta}_{\nu\mu}$ be the μ th argument (where μ = 1, 2, 3 . . .) of the Bessel function J_ν, for which $\frac{d J_\nu(\eta)}{d\eta} = 0$. The functions $Z_{\nu\mu}$ are given by

$$(1.13) \quad Z_{\nu\mu}(y) = J_\nu(y \, \bar{\eta}_{\nu\mu})$$

The functions $Z_{\nu\mu}$ fulfill the following differential equation

$$(1.14) \quad \frac{d^2}{dy^2}(Z_{\nu\mu}) + \frac{\frac{d}{dy}(Z_{\nu\mu})}{y} + \left(\bar{\eta}^2_{\nu\mu} - \frac{\nu^2}{y^2}\right) Z_{\nu\mu} = 0$$

There exists a relationship of orthogonality between the $Z_{\nu\mu}$, so that for $\mu \neq \nu$

$$(1.15) \quad \int_0^1 y \, Z_{\nu\mu} \, Z_{\nu\mu} \, dy = 0$$

Let us introduce

$$(1.16) \quad \int_0^1 y \, Z^2_{\nu\mu} \, dy = K_{\nu\mu}$$

A hypothesis for the perturbation velocity potential is chosen in such a way that the boundary conditions are fulfilled.

$$\varphi = \tfrac{1}{2} y^2 \Theta_0(x) + \sum_{\nu=1}^{\infty} \left(\tfrac{1}{\nu} y^\nu \Theta_\nu(x) \cos \nu\omega\right) + \sum_{\mu=1}^{\infty} \left(a_{0\mu}(x)\, Z_{0\mu}(y)\right) + \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} \left(a_{\nu\mu}(x)\, Z_{\nu\mu}(y) \cos \nu\omega\right) \tag{1.17}$$

The functions $a_{0\mu}(x)$ and $a_{\nu\mu}(x)$ ($\nu = 1, 2, 3 \ldots$) must be determined in such a way that the differential equations for φ (Equation (1.12) is fulfilled. Inserting this hypothesis into Equation (1.12), one obtains

$$\begin{aligned}
&-(\gamma+1)\left[\varphi_{Bx} \sum_{\mu=1}^{\infty} (a_{0\mu}'' Z_{0\mu}) + \varphi_{Bxx} \sum_{\mu=1}^{\infty} (a_{0\mu}' Z_{0\mu})\right] \\
&+ \sum_{\mu=1}^{\infty} a_{0\mu} \left(\frac{d^2 Z_{0\mu}}{dy^2} + \frac{1}{y} \frac{dZ_{0\mu}}{dy}\right) - (\gamma+1)\left[\tfrac{1}{2} y^2 (\varphi_{Bx} \Theta_0'' + \varphi_{Bxx} \Theta_0')\right] \\
&+ 2\Theta_0 - (\gamma+1)\left[\varphi_{Bx} \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} (a_{\nu\mu}'' Z_{\nu\mu} \cos \nu\omega) + \varphi_{Bxx} \right. \\
&\left. \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} (a_{\nu\mu}' Z_{\nu\mu} \cos \nu\omega)\right] + \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} a_{\nu\mu} \left(\frac{d^2 Z_{\nu\mu}}{dy^2} \cos \nu\omega \right. \\
&\left. + \frac{dZ_{\nu\mu}}{dy} \frac{\cos \nu\omega}{y} - \frac{\nu^2}{y^2} Z_{\nu\mu} \cos \nu\omega\right) - (\gamma+1)\left[\sum_{\nu=1}^{\nu=\infty} \frac{1}{\nu} y^\nu \right. \\
&\left. \cos \nu\omega (\varphi_{Bx} \Theta_\nu'' + \varphi_{Bxx} \Theta_\nu')\right] = -(\gamma+1)(\varphi_{Bx} \cdot \varphi_{Bxx})
\end{aligned} \tag{1.18}$$

All the terms of this equation which contain trigonometric terms of $\nu\omega$ (where $\nu = 1, 2, 3, 4$, etc.) must vanish separately.

For $\nu = 0$ one has the condition

$$\begin{aligned}
&-(\gamma+1)\left[\varphi_{Bx} \sum_{\mu=1}^{\infty} (a_{0\mu}'' Z_{0\mu}) + \varphi_{Bxx} \sum_{\mu=1}^{\infty} (a_{0\mu} Z_{0\mu})\right] \\
&+ \sum_{\mu=1}^{\infty} a_{0\mu} \left(\frac{d^2 Z_{0\mu}}{dy^2} + \frac{1}{y} \frac{dZ_{0\mu}}{dy}\right) - (\gamma+1)\left[\frac{y^2}{2} (\varphi_{Bx} \Theta_0'' + \varphi_{Bxx} \Theta_0')\right] \\
&= -2\Theta_0 - (\gamma+1)\, \varphi_{Bx} \varphi_{Bxx}
\end{aligned} \tag{1.19}$$

The conditions for $\nu = 1, 2, 3, \ldots$ are

$$\begin{aligned}
&-(\gamma+1)\left[\varphi_{Bx} \sum_{\mu=1}^{\infty} (a_{\nu\mu}'' Z_{\nu\mu}) + \varphi_{Bxx} \sum_{\mu=1}^{\infty} (a_{\nu\mu}' Z_{\nu\mu})\right] \\
&+ \sum_{\mu=1}^{\infty} a_{\nu\mu} \left(\frac{d^2 Z_{\nu\mu}}{dy^2} + \frac{1}{y} \frac{dZ_{\nu\mu}}{dy} - \frac{\nu^2}{y^2} Z_{\nu\mu}\right) \\
&= \left(\frac{\gamma+1}{\nu}\right) y^\nu (\varphi_{Bx} \Theta_\nu'' + \varphi_{Bxx} \Theta_\nu')
\end{aligned} \tag{1.20}$$

It can readily be seen from Equation (1.10) that

$$2\Theta_0 = -(\gamma+1)\, \varphi_{Bx} \varphi_{Bxx} \tag{1.21}$$

Further simplifications are made using Equation (1.14) with Equations (1.19) and (1.20), respectively.

$$(1.22)\quad -(\gamma+1)\sum_{\mu=1}^{\infty}\left[Z_{o\mu}\left(\Phi_{Bx}\,a_{o\mu}'' + \Phi_{Bxx}\,a_{o\mu}'\right)\right] - \sum_{\mu=1}^{\infty}\left[\bar{\eta}_{o\mu}^{2}\,Z_{o\mu}\,a_{o\mu}\right] = \frac{(\gamma+1)}{2}\,y^{2}\left(\Phi_{Bx}\,\Theta_o'' + \Phi_{Bxx}\,\Theta_o'\right)$$

$$(1.23)\quad -(\gamma+1)\sum_{\mu=1}^{\infty}\left[Z_{\nu\mu}\left(\Phi_{Bx}\,a_{\nu\mu}'' + \Phi_{Bxx}\,a_{\nu\mu}'\right)\right] - \sum_{\mu=1}^{\infty}\left[\bar{\eta}_{\nu\mu}^{2}\,Z_{\nu\mu}\,a_{\nu\mu}\right] = \frac{(\gamma+1)}{\nu}\,y^{\nu}\left(\Phi_{Bx}\,\Theta_\nu'' + \Phi_{Bxx}\,\Theta_\nu'\right)$$

The equations for the perturbation functions $a_{\nu\mu}$ are obtained by multiplying with the term $y\,Z_{\nu\mu}$ and integrating between $y = 0$ to $y = 1$. Because of the orthogonality relations for the Bessel functions, all with the terms $\int y\,Z_{\nu\mu}\,Z_{\nu\mu}\,dy$ drop out except for $\mu=\nu$. Inserting these relations in Equations (1.22) and (1.23) one obtains Equations (1.24) and (1.25), respectively.

$$(1.24)\quad I_{o\mu}\left[\Phi_{Bx}\,a_{o\mu}'' + \Phi_{Bxx}\,a_{o\mu}' + \frac{\bar{\eta}_{o\mu}^{2}}{\gamma+1}\,a_{o\mu}\right] = K_{o\mu}\,\frac{1}{2}\left(\Phi_{Bx}\,\Theta_o'' + \Phi_{Bxx}\,\Theta_o'\right)$$

where

$$(1.25)\quad K_{o\mu} = \int_0^1 Z_{o\mu}(y)\,y^{3}\,dy = \frac{2}{\bar{\eta}_{o\mu}^{2}}\,J_o(y\,\bar{\eta}_{o\mu})$$

$$(1.26)\quad I_{o\mu} = \int_0^1 Z_{o\mu}(y)\,y\,dy = \frac{1}{2}\,J_o^{2}(y\,\bar{\eta}_{o\mu})$$

(Derivation may be found in the Appendix E). Also,

$$(1.27)\quad I_{\nu\mu}\left[\Phi_{Bx}\,a_{\nu\mu}'' + \Phi_{Bxx}\,a_{\nu\mu}' + \frac{\bar{\eta}_{\nu\mu}^{2}}{\gamma+1}\,a_{\nu\mu}\right] = -K_{\nu\mu}\,\frac{1}{\nu}\left(\Phi_{Bx}\,\Theta_\nu'' + \Phi_{Bxx}\,\Theta_\nu'\right)$$

$$(1.28)\quad I_{\nu\mu} = \int_0^1 Z_{\nu\mu}(y)\,y\,dy = J_\nu^{2}(\bar{\eta}_{\nu\mu}\,y)$$

$$(1.29)\quad K_{\nu\mu} = \int_0^1 Z_{\nu\mu}(y)\,y^{\nu+1}\,dy = \frac{\nu}{\bar{\eta}_{\nu\mu}^{2}}\,J_\nu\left(\bar{\eta}_{\nu\mu}\,y\right)$$

From Equations (1.24) and (1.27), and the boundary conditions which will be given later, the values of $a_{\nu\mu}$ can be found. The resulting expression for the perturbation potential (Equation (1.17)) is written in a more convenient manner if one expresses the $1/2\ y^2$ and $\frac{1}{\nu}\,y^{\nu}$ by means of $Z_{o\mu}$ and $Z_{\nu\mu}$.

$$\frac{1}{2}\,y^{2} = \sum b_{o\mu}\,Z_{o\mu}(y)$$

$$\frac{1}{\nu}\,y^{\nu} = \sum b_{\nu\mu}\,Z_{\nu\mu}(y)$$

where $b_{o\mu} = \frac{\Theta_o}{2} \frac{K_{o\mu}}{I_{o\mu}}$

Let us introduce

(1.30) $\quad b_{o\mu} + a_{o\mu} = a^*_{o\mu}$

(1.31) $\quad b_{\nu\mu} + a_{\nu\mu} = a^*_{\nu\mu}$

The perturbation potential is then based entirely in terms of the functions $Z_{\nu\mu}$ so that

(1.32) $$\bar{\Phi} = \sum_{\mu=0}^{\infty} \left[a^*_{o\mu} Z_{o\mu} \right] + \sum_{\mu=0}^{\infty} \sum_{\nu=1}^{\infty} \left[a_{\nu\mu} Z_{\nu\mu} \right]$$

One can then write the differential equations for $a^*_{\nu\mu}$ by the introduction of Equations (1.30) and (1.31) into Equations (1.24) and (1.27).

(1.33) $$\Phi_{B_x} a^{*\,\prime\prime}_{o\mu} + \Phi_{B_{xx}} a^{*\,\prime}_{o\mu} + \frac{\bar{\eta}^2_{o\mu}}{\gamma+1} a^*_{o\mu} = \frac{K_{o\mu}}{2 I_{o\mu}} \frac{\bar{\eta}^2_{o\mu}}{\gamma+1} \Theta_o$$

(1.34) $$\Phi_{B_x} a^{*\,\prime\prime}_{\nu\mu} + \Phi_{B_{xx}} a^{*\,\prime}_{\nu\mu} + \frac{\bar{\eta}^2_{\nu\mu}}{\gamma+1} a^*_{\nu\mu} = \frac{K_{\nu\mu}}{2 I_{\nu\mu}} \frac{\bar{\eta}^2_{\nu\mu}}{\gamma+1} \Theta_\nu$$

φ and its first derivatives with respect to x are continuous. Therefore, according to this relation, $a^*_{\nu\mu}$ and their first derivatives are continuous.

One might ask why the hypothesis (Equation (1.32)) was not used from the beginning instead of Equation (1.17). The reason is that the second derivative with respect to y does not converge (because of the character of the function y^ν). Therefore, all of the considerations regarding the fulfillment of the differential equation for φ would have been meaningless.

Method of Solution

The solution of the perturbation functions $a^*_{\nu\mu}$ for the differential equations (1.33) and (1.34) is found for any duct, within the defined limitations, by means of numerical integration. For suitable configurations, analytical solutions can be found. In some cases, a piecewise analytical solution may be satisfactory. In any case, the pressure distribution along the axis is influenced only by the expressions with $\nu = 0$, since the nature of higher order Bessel functions is such that there is no effect along the axis.

Deviation from a parallel flow are found from φ_y. The only expressions which are of importance along the axis are those for $\nu = 0$ and $\nu = 2$. Actually, an infinite number of terms are needed to describe the flow completely, but the higher terms in μ and ν are very small and are of influence only close to the wall.

Conditions for the Solution of Equations (1.33) and (1.34)

Equations (1.33) and (1.34) are of the second order, and consequently the solutions must be determined by two boundary conditions. The first condition is obvious. With the assumption of a subsonic duct extending to negative infinity, one may assume that the values of the perturbation terms $\alpha^*_{\nu\mu}$ should vanish as one approaches negative infinity.

The character of the second condition is less obvious. One may first ask at which point such a condition might be given. Certainly it cannot be a point in the supersonic region, for the fulfillment of the condition will affect the entire solution, including points upstream of the point where the boundary condition is given. Since this would violate the law of the forbidden signals, the location of the other condition should be expected to be in the subsonic region. Actually, mathematical reasoning leads to the location of the boundary condition at the narrowest cross-section. If one rewrites Equations (1.33) and (1.34) in such a way that the coefficient of the highest derivative of the term $\alpha^*_{\nu\mu}$ is one, the coefficients of the other terms become singular at the sonic velocity. The solution of the differential equation may be singular at any point where these coefficients are singular.

Consider now only the solutions of the homogeneous part of the differential equation, since the influence of change in the arbitrary constants and consequently, the influence of the boundary condition is limited to that part. Let the basic flow velocity be represented by $\varphi_{B_x} = x^{\bar{\alpha}}$. Then the two linearly independent solutions of the homogeneous differential equation are represented by

$$(1.35) \qquad \alpha^*_{\nu\mu} = P(x)$$

$$(1.36) \qquad \alpha^*_{\nu\mu} = x^{1-\bar{\alpha}}\, P(x)$$

where P denotes a power series of x, which starts with the absolute term.

As the term φ_{B_x} approaches zero (at the throat) we have $\bar{\alpha} > 0$. Then the expression $\alpha^*_{\nu\mu}$, from Equation (1.36) will tend to infinity as $x \to 0$. Since this is physically impossible, the second condition is that remains finite at the throat.

To summarize the boundary conditions, the flow deviation expressions $\alpha^*_{\nu\mu}$ must be finite at x = 0 and zero as $x \to -\infty$.

SECTION III

APPLICATIONS

Flow Through A Circular Subsonic-Supersonic Duct

We apply this theory to an axially-symmetric transonic duct configuration. This duct is similar to the modified 10-Foot Wind Tunnel test section

at Wright-Patterson Air Force Base. The duct is first cylindrical (Section A, Figures 4 and 5) followed by a throat region of parabolic profile (Section B), and finally the duct expands conically (Section C). The configuration is geometrically determined by the subsonic junction position x_0 , the radius of curvature R, and the angle of expansion α It is desired to trace the magnitudes of the x-velocity deviations from a uniform, parallel flow through the duct channel.

In the differential equation (1.33) for $a^*_{0\mu}$, the shape of the duct expresses itself by the function $\Phi_B(x)$, which occurs in the coefficients of $a^{*\prime\prime}_{0\mu}$ and $a^{*\prime}_{0\mu}$, and furthermore, in the function Θ_0 which occurs in the righthand term.

In Section A, the duct is cylindrical and consequently the radius, slope, and Φ_{Bx} terms are

(2.1) $r = r_a = 1 + \dfrac{x_0^2 r_0}{2R}$

(the value of r_a can be seen from the subsequent formula (2.4)).

(2.2) $\Theta_0 = 0$

(2.3) $\Phi_{Bx} = \left(\dfrac{4}{\gamma+1}\, C\right)^{\frac{1}{2}}$ (for section A)

The profile of Section B is an approximation for a circular arc with the radius of curvature R. When approximated by a parabola with the vertex at the minimum section of the duct, the shape is expressed by:

(2.4) $r = 1 + \dfrac{x^2 r_0}{2R}$

(2.5) $\Theta_0 = \dfrac{x\, r_0}{R}$

(2.6) $\Phi_{Bx} = x\left(\dfrac{2}{\gamma+1}\,\dfrac{r_0}{R}\right)^{\frac{1}{2}}$ (for section B)

Section C starts tangential to the parabolic Section B and expands conically at the angle α . The shape terms are

(2.7) $r = 1 + \left(x\alpha - \dfrac{R\alpha^2}{2r_0}\right)$

(2.8) $\Theta_0 = \alpha$

(2.9) $\Phi_{Bx} = \left(\dfrac{4\alpha}{\gamma+1}\left(x - \dfrac{R\alpha}{2r_0}\right)\right)$ (for section C)

These values inserted into the differential equation (1.28) lead to the specific differential equation for each section. 1.33

Section A

(2.10) $\Phi_{Bx}\, a^{*\prime\prime}_{0\mu} + \dfrac{\bar{\eta}^2_{0\mu}}{\gamma+1}\, a^*_{0\mu} = 0$

Section B

$$(2.11)\quad x\,\alpha^{*\prime\prime}_{o\mu} + \alpha^{*\prime}_{o\mu} + E_{1\mu}\,\alpha^{*}_{o\mu} = E_{2\mu}\,x$$

The constants are:

$$(2.12)\quad E_{1\mu} = \frac{\frac{\bar{\eta}^2_{o\mu}}{\gamma+1}}{\left(\frac{2}{\gamma+1}\,\frac{r_o}{R}\right)^{\frac{1}{2}}}$$

$$(2.13)\quad E_{2\mu} = \frac{r_o}{2R}\,\frac{\frac{\bar{\eta}^2_{o\mu}}{\gamma+1}\,\frac{K_{o\mu}}{I_{o\mu}}}{\left(\frac{2}{\gamma+1}\,\frac{r_o}{R}\right)^{\frac{1}{2}}}$$

Section C

$$(2.14)\quad \left[\frac{4\alpha}{\gamma+1}\left(x-\frac{R\alpha}{2r_o}\right)\right]^{\frac{1}{2}}\alpha^{*\prime\prime}_{o\mu} + \frac{1}{2}\left(\frac{4\alpha}{\gamma+1}\right)^{\frac{1}{2}}\left(x-\frac{R\alpha}{2r_o}\right)^{-\frac{1}{2}}\alpha^{*\prime}_{o\mu} + \frac{\bar{\eta}^2_{o\mu}}{\gamma+1}\,\alpha^{*}_{o\mu} = \frac{K_{o\mu}}{2I_{o\mu}}\,\frac{\bar{\eta}_{o\mu}}{\gamma+1}\,\alpha$$

The solutions of the differential equations (2.10), (2.11), and (2.14) will now be determined. The complete solution, in each case, is the sum of the general solution of the homogeneous portion and a particular solution of the non-homogeneous equation. The constants will be determined later.

Section A

The solution of Equation (2.10) is

$$(2.15)\quad \alpha^{*}_{o\mu} = C_{1\mu}\,e^{\left(-\frac{\bar{\eta}^2_{o\mu}}{\varphi_{ex}(\gamma+1)}\right)^{\frac{1}{2}}x} + C_{2\mu}\,e^{-\left(-\frac{\bar{\eta}^2_{o\mu}}{\varphi_{ex}(\gamma+1)}\right)^{\frac{1}{2}}x}$$

Section B

The particular solution of Equation (2.11) is

$$(2.16)\quad \alpha^{*}_{o\mu} = E_{3\mu} + E_{4\mu}\,x$$

where

$$(2.17)\quad E_{3\mu} = -\frac{r_o}{2R}\,\frac{K_{o\mu}}{I_{o\mu}}\,\frac{\left(\frac{2}{\gamma+1}\,\frac{r_o}{R}\right)^{\frac{1}{2}}}{\frac{\bar{\eta}^2_{o\mu}}{\gamma+1}}$$

$$(2.18)\quad E_{4\mu} = \frac{r_o}{2R}\,\frac{K_{o\mu}}{I_{o\mu}}$$

The homogeneous part of Equation (2.11), through a simple transformation, is of the form

$$(2.19)\quad y'' + \frac{1}{x}\,y' - \left[\frac{1}{x} + \left(\frac{p}{2x}\right)^2\right]y = 0$$

the solution[1] of which is

$$(2.20)\quad y = Z_p\left[2\lambda\, x^{\frac{1}{2}}\right]$$

The term Z_p denotes any linear combination of Bessel functions of order p. The general solution of the homogeneous part of Equation (2.11) then is such that the complete solution becomes

$$(2.21)\quad a^*_{o\mu} = E_{3\mu} + E_{4\mu}x + C_{3\mu}J_o\left[2\,E_{1\mu}^{\frac{1}{2}}x^{\frac{1}{2}}\right] + C_{4\mu}N_o\left[2\,E_{1\mu}^{\frac{1}{2}}x^{\frac{1}{2}}\right]$$

Section C

A particular solution of Equation (2.14) in Section C is given by

$$(2.22)\quad a^*_{o\mu} = \frac{K_{o\mu}}{2\,I_{o\mu}}\,\alpha$$

In order to find the general solution of the homogeneous part of Equation (1.28), the transformations $u = \left(x - \frac{R\alpha}{2r_o}\right)^{\frac{1}{2}}$ and $a^*_{o\mu}(x) = q(u)$ are carried out. The differential equation is then written

$$(2.23)\quad \frac{1}{4u}\left(q'' - \frac{q}{u}\right) + \frac{1}{2u}\left(\frac{q}{2u}\right) + E_{5\mu}\,q = 0$$

where

$$(2.24)\quad E_{5\mu} = \frac{\bar{\eta}_{o\mu}^{2}}{2\left[\alpha(\gamma+1)\right]^{\frac{1}{2}}}$$

Equation (2.23) has the form

$$(2.25)\quad y + bxy = 0$$

where the solution[2] is given as

$$(2.26)\quad y = x^{\frac{1}{2}}\,Z_{\frac{1}{3}}\left[\tfrac{2}{3}\,b^{\frac{1}{2}}x^{\frac{3}{2}}\right]$$

Consequently, the solution of Equation (2.23) is

$$(2.27)\quad q = u^{\frac{1}{2}}\,Z_{\frac{1}{3}}\left[\tfrac{4}{3}E_{5\mu}^{\frac{1}{2}}u^{\frac{3}{2}}\right]$$

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[1] Jahnke, E. and Emde, F., Tables of Functions. Fourth Edition. Dover Publications, 1945, p. 147.

[2] Ibid., p. 147.

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(2.35)

$$C_{6\mu} = \frac{E_{5\mu}^{\frac{1}{2}} J_{-\frac{2}{3}}(e_1)\left[E_{3\mu} + E_{4\mu} x_1 + C_{3\mu} J_0(e_0) - \frac{K_{0\mu}}{2 I_{0\mu}} \alpha\right] - \left(x_1 - \frac{R\alpha}{2r_0}\right)^{\frac{1}{4}} J_{\frac{1}{3}}(e_1)\left[E_{4\mu} - J_1(e_0)\frac{C_{3\mu} E_{1\mu}^{\frac{1}{2}}}{X_1^{\frac{1}{2}}}\right]}{\left(X_1 - \frac{R\alpha}{2r_0}\right)^{\frac{1}{4}} E_{5\mu}^{\frac{1}{2}}\left[J_{\frac{1}{3}}(e_1) J_{\frac{2}{3}}(e_1) + J_{-\frac{1}{3}}(e_1) J_{-\frac{2}{3}}(e_1)\right]}$$

where

$$e_0 = 2 E_{1\mu}^{\frac{1}{2}} X_0^{\frac{1}{2}}$$

By the use of the previous formulae, the expressions $a^*_{0\mu}$ is determined completely. The quantities X_0, R, and α which determine the duct geometry are determined by the specific problem under consideration.

Discussion of Results

In the previous formulae the quantities X_0, R, and α are written in the general form. This, of course, enables us to determine the solutions for different configurations.

In the following analysis the values which correspond to the Wright-Patterson Air Force Base 10-Foot Wind Tunnel will be inserted. Before this is done, the effect of the choice of the subsonic junction position X_0 is studied. Most investigations of a supersonic flow are made disregarding any influences of the subsonic portion of the duct, since there is no theoretical basis for its consideration. It is of interest to determine some numerical values of the subsonic effects. Therefore, a flow pattern determination will be made of a configuration which corresponds to the supersonic part of the 10-Foot Wind Tunnel, and three different subsonic junction positions.

Accordingly, the duct shape gives

$$R = 50\, r_0 \qquad \alpha = .15 \text{ degrees}$$

$$x_0 = -0.10,\ -0.20,\ -0.40$$

The values of $\bar{\eta}_{0\mu}$ are the Bessel function arguments for which $\frac{d}{d\eta}[J_0(\eta)] = 0$ and are

$$\bar{\eta}_{01} = 3.832$$

$$\bar{\eta}_{02} = 7.015$$

$$\bar{\eta}_{03} = 10.1735$$

The values of $E_{1\mu}$, $E_{2\mu}$, $E_{3\mu}$, and $E_{4\mu}$ are determined by Equations (2.12), (2.13), (2.17), and (2.18), respectively. The junction constants $C_{3\mu}$, $C_{5\mu}$, and $C_{6\mu}$ are obtained by means of Equations (2.33), (2.34), and (2.35), respectively.

Of special interest is the velocity distribution along the axis of the duct. The x-velocity terms found from one-dimensional theory are given by Equations (2.3), (2.6) and (2.9). The additional x-velocity along the axis caused by the perturbation potential $\bar{\Phi}_x$ are found from Equation (1.32). Since $J_0(0)=1$, one obtains

$$(2.36) \quad \bar{\Phi}_x = \sum_{\mu} a^{*\prime}_{0\mu}$$

Figure 6 shows the $a^{*\prime}_{01}$ values through Section B for $x_0 = -.01, -.02, -.04, -\infty$ In spite of appreciable initial flow deviations at some distance upstream from the throat, the effects are hardly noticeable at the minimum cross section and in the supersonic region. A corresponding behavior is found for the terms $a^{*\prime}_{02}$, $a^{*\prime}_{03}$, etc. Accordingly, even extreme changes in the subsonic region are without importance for the supersonic region.

Figure 8 shows the velocity distribution along the axis according to the one-dimensional theory, and a velocity which includes the corrections given by a^{*}_{01}, a^{*}_{02}, and a^{*}_{03}. Beyond x = 0.5 the velocity distribution is rather smooth, showing the present configuration to be satisfactory. The maximum velocity deviations from the one-dimensional theory have a magnitude corresponding to a Mach number of about .004. The velocity deviation at about x = 0.4 is rather large, but this is to be expected from the following reasoning. Point M (Figure 3) is considered to be the junction between Sections B and C. The effect of the change of shape at this point propagates to the axis along the Mach lines MN. Upstream of MN the flow, especially along the axis, corresponds to the parabolic surfaces. Consequently, any change in pressure distribution caused by the transition is felt only downstream of point B. The one-dimensional theory (Figure 8, curve φ_{Bx}) shows the influence of the junctions to be right at point A. The superposition of the $\bar{\Phi}_x$ value shows the occurrence of the change at x = .38. The x-coordinate of Point N in Figure 3, computed by an exact relationship for the Mach angle, is x = .387. Figure 8, curve $(\varphi_{Bx} + \bar{\Phi}_x)$ shows what might be expected from a more general reasoning. The functions $a^{*\prime}_{01}$, $a^{*\prime}_{02}$, and $a^{*\prime}_{03}$ are given in Figure 7.

A Remark on the Application to a Non-Symmetric Configuration

Although no examples of the application for an unsymmetric flow are carried out, an idea of the procedure will be given. The shape of the duct must be expressed according to Equation (1.6). For instance, one must carry out a harmonic analysis of the function $r(x,\omega)$ with respect to ω for various values of x. This can be done by a numerical or analytical evaluation of the integrals (1.6a) and (1.6b), or by means of a harmonic analyzer. Then the values of Θ_ν are found from Equation (1.9). Following this preparation, the procedure can be carried out in the same way as for an axially symmetric duct.

Solution with φ_{B_x} as A Function of x^*

A solution of the differential equation is determined for the coefficient φ_{B_x} as a power of the longitudinal position. The solution and its derivation will be found in the Appendix (Section IV, Part F).

Concluding Remarks

A method for the analysis of transonic flow through a nearly circular duct with only gradual changes in cross-section has been derived. One is finally lead to the solutions of ordinary linear differential equations. (Equations (1.33) and (1.34)). These solutions can be carried out numerically, or even analytically in many cases.

Application of this method was made to a configuration similar to the Wright-Patterson Air Force Base 10-Foot Wind Tunnel. A velocity distribution through the duct was determined and the disturbances caused by the variation in duct shapes were computed. The velocity profile for the configuration is satisfactory. The subsonic influence in this case is small enough that the usual consideration of no subsonic disturbances propagated into the supersonic region is justified.

SECTION IV

APPENDIX A

Derivation of the Potential Equation

It follows from the vorticity theorems that, with certain conditions for the initial flow which are fulfilled in most practical cases, there exists a velocity potential $\Phi(\bar{x}, \bar{y}, \omega)$, where the coordinates are shown in Figure 1. (That is, the velocity vector can be expressed as the gradient of a scalar function Φ). If one chooses an arbitrary function, $\Phi(\bar{x}, \bar{y}, \omega)$ and determines the velocity components accordingly; i.e.,

$$V_{\bar{x}} = \Phi_{\bar{x}} \tag{3.1}$$

$$V_{\bar{y}} = \Phi_{\bar{y}} \tag{3.2}$$

$$V_{\omega} = \frac{\Phi_{\omega}}{\bar{y}} \tag{3.3}$$

and furthermore, determines the pressure from Bermoulli's equation, then Euler's equations of motion are automatically fulfilled. Therefore, if a velocity potential exists, we have to check only the condition of continuity. The continuity equation is given by

$$\frac{\partial}{\partial \bar{x}}(\rho V_{\bar{x}}) + \frac{1}{\bar{y}}\frac{\partial}{\partial \bar{y}}(\rho \bar{y} V_{\bar{y}}) + \frac{1}{\bar{y}}\frac{\partial}{\partial \omega}(\rho V_{\omega}) = 0 \tag{3.4}$$

If one inserts Equations (3.1), (3.2), and (3.3) into Equation (3.4)

$$\frac{\partial}{\partial \bar{x}}(\rho \Phi_{\bar{x}}) + \frac{\partial}{\partial \bar{y}}(\rho \Phi_{\bar{y}}) + \frac{\partial}{\partial \omega}\left(\rho \frac{\Phi_{\omega}}{\bar{y}}\right) = 0 \tag{3.5}$$

The derivatives of ρ are found from Bermoulli's equation for three-dimensional flow

$$i + \frac{1}{2}\left(\Phi_{\bar{x}}^2 + \Phi_{\bar{y}}^2 + \frac{\Phi_{\omega}^2}{\bar{y}^2}\right) = \text{constant} \tag{3.6}$$

where i denotes the enthalpy, so that

$$i = \frac{\alpha^2}{\gamma+1}$$

From the isentropic flow relations, since $i = C_p T$ for perfect gases.

$$\frac{i}{i_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma-1}$$

Consequently, one writes

$$\frac{di}{i} = (\gamma-1)\frac{d\rho}{\rho}$$

Or

$$di = a^2 \frac{d\rho}{\rho} \tag{3.7}$$

If Equation (3.6) is differentiated, and Equation (3.7) is inserted the resulting expression is

$$a^2 \frac{\partial \rho}{\rho} + \phi_{\bar{x}}\, d\phi_{\bar{x}} + \phi_{\bar{y}}\, d\phi_{\bar{y}} + \frac{\phi_\omega}{\bar{y}}\, d\left(\frac{\phi_\omega}{\bar{y}}\right) = 0$$

Or

(3.8) $$d\rho = -\frac{\rho}{a^2}\left(\phi_{\bar{x}}\, d\phi_{\bar{x}} + \phi_{\bar{y}}\, d\phi_{\bar{y}} + \frac{\phi_\omega}{\bar{y}}\, d\left(\frac{\phi_\omega}{\bar{y}}\right) = 0\right.$$

Writing Equation (3.8) in detail, one obtains

(3.8a) $$\frac{\partial \rho}{\partial \bar{x}} = -\frac{\rho}{a^2}\left(\phi_{\bar{x}}\, \phi_{\bar{x}\bar{x}} + \phi_{\bar{y}}\, \phi_{\bar{x}\bar{y}} + \frac{\phi_\omega \phi_{\bar{x}\omega}}{\bar{y}^2}\right)$$

(3.8b) $$\frac{\partial \rho}{\partial \bar{y}} = -\frac{\rho}{a^2}\left(\phi_{\bar{x}}\, \phi_{\bar{x}\bar{y}} + \phi_{\bar{y}}\, \phi_{\bar{y}\bar{y}} + \frac{\phi_\omega^2}{\bar{y}^3} + \frac{\phi_\omega\, \phi_{\omega\bar{y}}}{\bar{y}^2}\right.$$

(3.8c) $$\frac{\partial \rho}{\partial \omega} = -\frac{\rho}{a^2}\left(\phi_{\bar{x}}\, \phi_{\bar{x}\omega} + \phi_{\bar{y}}\, \phi_{\bar{y}\omega} + \frac{\phi_\omega\, \phi_{\omega\omega}}{\bar{y}^2}\right)$$

Now, if the partial derivatives of ρ from Equations (3.8a), (3.8b), and (3.8c) are inserted in Equation (3.5), one obtains the expression

(3.9) $$-\frac{\rho}{a^2}\left(\phi_{\bar{x}}\, \phi_{\bar{x}\bar{x}} + \phi_{\bar{y}}\, \phi_{\bar{x}\bar{y}} + \frac{\phi_\omega\, \phi_{\bar{x}\omega}}{\bar{y}^2}\right)\phi_{\bar{x}} + \rho\, \phi_{\bar{x}\bar{x}}$$
$$-\frac{\rho}{a^2}\left(\phi_{\bar{x}}\, \phi_{\bar{x}\bar{y}} + \phi_{\bar{y}}\, \phi_{\bar{y}\bar{y}} + \frac{\phi_\omega^2}{\bar{y}^3} + \frac{\phi_\omega\, \phi_{\omega\bar{y}}}{\bar{y}^2}\right)\phi_{\bar{y}} + \frac{\rho\, \phi_{\bar{y}}}{\bar{y}}$$
$$+ \rho\, \phi_{\bar{y}\bar{y}} - \frac{\rho}{a^2}\left(\frac{\phi_{\bar{x}}\, \phi_{\bar{x}\omega}}{\bar{y}} + \frac{\phi_{\bar{y}}\, \phi_{\bar{y}\omega}}{\bar{y}} + \frac{\phi_\omega\, \phi_{\omega\omega}}{\bar{y}^3}\right)\frac{\phi_\omega}{\bar{y}} + \rho\frac{\phi_{\omega\omega}}{\bar{y}^2} = 0$$

If one divides Equation (3.9) by ρ and collects the terms, one obtains the final equation

(3.10) $$\phi_{\bar{x}\bar{x}}\left(1 - \frac{\phi_{\bar{x}}^2}{a^2}\right) - 2\,\phi_{\bar{x}\bar{y}}\left(\frac{\phi_{\bar{x}}\, \phi_{\bar{y}}}{a^2}\right) + \phi_{\bar{y}\bar{y}}\left(1 - \frac{\phi_{\bar{y}}^2}{a^2}\right)$$
$$+ \phi_{\omega\omega}\left(1 - \frac{\phi_\omega^2}{\bar{y}^4 a^2}\right) - 2\,\frac{\phi_{\bar{x}}\, \phi_\omega\, \phi_{\bar{x}\omega}}{\bar{y}^2 a^2} - 2\,\frac{\phi_{\bar{y}}\, \phi_\omega\, \phi_{\bar{y}\omega}}{\bar{y}^2 a^2} + \frac{\phi_{\bar{y}}}{\bar{y}} - \frac{\phi_\omega^2\, \phi_{\bar{y}}}{\bar{y}^3 a^2} = 0$$

This is the complete potential equation for three-dimensional flow.

APPENDIX B

Transonic Law of Similarity

The general three-dimensional equation for a steady potential flow is given by:

$$(3.11)\quad \Phi_{\bar{x}\bar{x}}\left(1-\frac{\Phi_{\bar{x}}^2}{a^2}\right)-2\frac{\Phi_{\bar{x}\bar{y}}(\Phi_{\bar{x}}\Phi_{\bar{y}})}{a^2}+\Phi_{\bar{y}\bar{y}}\left(1-\frac{\Phi_{\bar{y}}^2}{a^2}\right)+\Phi_{\omega\omega}\left(1-\frac{\Phi_\omega^2}{\bar{y}^4a^2}\right)$$
$$-2\frac{\Phi_{\bar{x}}\Phi_\omega\Phi_{\bar{x}\omega}}{\bar{y}^2a^2}-2\frac{\Phi_{\bar{y}}\Phi_\omega\Phi_{\bar{y}\omega}}{\bar{y}^2a^2}+\frac{\Phi_{\bar{y}}}{\bar{y}}-\frac{\Phi_\omega^2\Phi_{\bar{y}}}{\bar{y}^3a^2}=0$$

The velocity of sound a is obtained from Bernoulli's equation.

$$(3.12)\quad a^2=\frac{\gamma+1}{2}a^{*2}-\frac{\gamma-1}{2}w^2$$

Where the total stream velocity:

$$w=\left(\Phi_{\bar{x}}^2+\Phi_{\bar{y}}^2+\frac{\Phi_\omega^2}{\bar{y}^2}\right)^{\frac{1}{2}}$$

Let us consider a family of flow patterns, where each flow pattern is characterized by a specific value of a parameter z . Let the velocity potential be given by:

$$(3.13)\quad \Phi=a^*\left(\bar{x}+z\varphi(\tilde{x},\tilde{y},\omega)+z^2\varphi(\tilde{x},\tilde{y},\omega)+\ldots\ldots\right)$$

where

$$(3.14)\quad \tilde{x}=\bar{x}$$

$$(3.15)\quad \tilde{y}=\bar{y}\;z^{\frac{1}{2}}$$

This hypothesis is justified, by the fact that its introduction into the differential equation for Φ , with the limiting process $z\to 0$, results into a non-trivial equation for φ . Indeed substitution of Equations (3.12), (3.13), (3.14), and (3.15) into the potential equation (Equation (3.11)) gives

$$(3.16)\quad a^*z\,\varphi_{\tilde{x}\tilde{x}}\left(1-\frac{(a^*+a^*z\varphi_{\tilde{x}})^2}{a^2}\right)-2a^*z^{\frac{3}{2}}\varphi_{\tilde{x}\tilde{y}}\left(\frac{a^{*2}(1+z\varphi_{\tilde{x}})z^{\frac{3}{2}}\varphi_{\tilde{y}}}{a^2}\right)$$
$$+a^*z^2\varphi_{\tilde{y}\tilde{y}}\left(1-\frac{(a^*z^{\frac{3}{2}}\varphi_{\tilde{y}})^2}{a^2}\right)+\frac{a^*z^2\varphi_{\tilde{y}}}{\tilde{y}}+\frac{a^*z^2\varphi_{\omega\omega}}{\tilde{y}^2}=0$$

The value of the sonic velocity as given in Equation (3.12) dan be written in terms of a^* and the velocity potential, and is finally expressed as

$$(3.17)\quad a^2=a^{*2}-(\gamma-1)(z\varphi_{\tilde{x}})a^*$$

1. The $\bar{x}$ coordinates of corresponding points are the same

2. The y coordinates of corresponding points are proportional to $z^{-\frac{1}{2}}$

3. The difference between the x velocity and the sonic velocity is proportional to z

4. The difference between the pressure and the sonic pressure is proportional to z

5. The deviation of the streamlines from lines of constant y and ω is proportional to $z^{\frac{1}{2}}$

APPENDIX C

Boundary Condition

Along a body at rest the velocity vector is parallel to the surface; that is, the component of flow normal to the surface is zero. To express the relation, let the unit vectors in the direction of increasing x, y and ω coordinates be U_x, U_y, and U_ω, respectively. The total velocity vector then is expressed by

$$(3.24)\quad a^*\left[(1+\varphi_x)U_x + \varphi_y U_y + \frac{\varphi_\omega}{y} U_\omega\right]$$

The vector component normal to the duct surface is given by

$$(3.25)\quad -\frac{\partial r}{\partial x}U_x - \frac{\partial r}{\partial \omega\, r}U_\omega + U_y = 0$$

The scalar product of these two terms is the component of the velocity vector normal to the surface. According to the boundary condition, this component is zero. Then one obtains

$$(3.26)\quad -a^*(1+\varphi_x)\frac{\partial r}{\partial x} + a^*\varphi_y - \frac{a^*\varphi_\omega}{r^2}\frac{\partial r}{\partial \omega} = 0$$

The assumption that φ is a small quantity implies that the slope of the duct and surface is gradual, and consequently $\frac{\partial r}{\partial x}$ and $\frac{\partial r}{\partial \omega}$ are small quantities.

Therefore, the expressions $\varphi_x \frac{\partial r}{\partial x}$ and $\varphi_\omega \frac{\partial r}{\partial \omega}$ are of second order, and can be neglected in Equation (3.26). This results in

$$(3.27)\quad -\frac{\partial r}{\partial x} + \varphi_y = 0$$

APPENDIX D

Derivation of the Basic Flow Velocity

The basic flow velocity φ_{B_x} is the velocity obtained from one-dimensional flow theory, and is expressed as a function of the duct radius at the cross section under consideration.

The continuity equation is written

$$(3.28)\quad r^2 \pi \rho V = r^{*2} \pi \rho^* a^*$$

where V is the absolute value of the velocity. Since ρV is a function of V, it can be developed with respect to (V - a*). One then obtains

$$(3.29)\quad \rho V = \rho^* a^* - \frac{\rho^*}{a^*}(\gamma+1)\frac{(V-a^*)^2}{2}$$

If one introduces the relation

$$(3.30)\quad \Delta r = r - r_0$$

the continuity equation is rewritten as

$$\left[\rho^* a^* - \frac{\rho^*}{a^*}(\gamma+1)\left(\frac{V-a^*}{2}\right)^2\right]\pi(r_0+\Delta r)^2 = \rho^* a^* \pi r_0^2$$

If r is small, the higher order terms can be neglected so that

$$(3.31)\quad (V-a^*) = \left[\frac{4a^{*2}}{\gamma+1}\frac{\Delta r}{r_0}\right]^{\frac{1}{2}}$$

Since $\varphi_{B_x} = \frac{V}{a^*} - 1$, the basic flow velocity is expressed by

$$(3.32)\quad \varphi_{B_x} = \left[\frac{4}{\gamma+1}\frac{\Delta r}{r_0}\right]^{\frac{1}{2}}$$

$\phi_x = \phi_{B_x} + \bar{\phi}_x$

APPENDIX E

Analytical Representation of the Constants $I_{o\mu}$, $K_{o\mu}$, $I_{\nu\mu}$, and $K_{\nu\mu}$

An analytical representation is sought for the integrals that occur in the expressions $I_{o\mu}$ and $K_{o\mu}$. The first term $I_{o\mu}$ is

$$(3.33)\quad I_{o\mu} = \int_0^1 y\, Z_{o\mu}^2(y)\,dy = \int_0^1 y\, J_0^2(\bar{\eta}_{o\mu} y)\,dy$$

Integrating[3]

$$(3.34)\quad I_{o\mu} = \frac{y^2}{2}\left[J_0^2(\bar{\eta}_{o\mu} y) - J_1(\bar{\eta}_{o\mu} y)\,J_{-1}(\bar{\eta}_{o\mu} y)\right]_{y=0}^{y=1}$$

[3] Ibid., p. 145

Then[4] since

$$J_{-1}(\bar{\eta}_{o\mu}y) = J_1(\bar{\eta}_{o\mu}y)$$

(3.35) $$I_{o\mu} = \frac{1}{2}\left[J_o^2(\bar{\eta}_{o\mu}y) + J_1^2(\bar{\eta}_{o\mu}y)\right]$$

The values for the arguments of the Bessel function $J_o(\bar{\eta}_{o\mu}y)$ were chosen so that $\frac{dJ_o(\eta)}{d\eta} = 0$. However, it is also true that $\frac{dJ_o(\eta)}{d\eta} = J_1(\eta)$, so that Equation (3.35) simplifies to the analytical expression

(3.36) $$I_{o\mu} = \frac{1}{2}J_o^2(\bar{\eta}_{o\mu}y)$$

The term $K_{o\mu}$ is defined by

(3.37) $$K_{o\mu} = \int_0^1 y^3 Z_{o\mu}(y)\,dy = \int_0^1 y^3 J_o(\bar{\eta}_{o\mu}y)\,dy$$

If one denotes the argument by $u = (\bar{\eta}_{o\mu}y)$ Equation (3.37) is rewritten as

$$K_{o\mu} = \int_0^{\bar{\eta}_{o\mu}} \frac{u^3}{\bar{\eta}_{o\mu}^3} J_o(u)\,\frac{du}{\bar{\eta}_{o\mu}}$$

or simply

(3.38) $$K_{o\mu} = \frac{1}{\bar{\eta}_{o\mu}^4}\int_0^{\bar{\eta}_{o\mu}} u^3 J_o(u)\,du$$

An integration[5] gives the relationship

(3.39) $$K_{o\mu} = \frac{1}{\bar{\eta}_{o\mu}^4}\left[u^3 J_1(u) - 2\int_0^{\bar{\eta}_{o\mu}} u^2\, J_1(u)\,du\right]$$

or

(3.40) $$K_{o\mu} = \frac{1}{\bar{\eta}_{o\mu}^4}\left[u^3 J_1(u) + 2u^2 J_o(u) - 4u\,J_1(u)\right]_0^{\eta_{o\mu}}$$

If the Equation (3.40) is rewritten in terms of y, and the limits of y = 0 and y = 1 are inserted, the new relation becomes

(3.41) $$K_{o\mu} = \frac{1}{\bar{\eta}_{o\mu}^4}\left[2\bar{\eta}_{o\mu}^2 J_o(\bar{\eta}_{o\mu}y) + (\bar{\eta}_{o\mu}^3 - 4\bar{\eta}_{o\mu})J_1(\bar{\eta}_{o\mu}y)\right]$$

Since $\frac{dJ_o(\eta)}{d\eta} = J_1(\eta) = 0$ the $K_{o\mu}$ is reduced to

(3.42) $$K_{o\mu} = \frac{2}{\bar{\eta}_{o\mu}^2} J_o(\bar{\eta}_{o\mu}y)$$

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4 Ibid., p. 145

5 Ibid., p. 145

The term $I_{\nu\mu}$ is defined as

$$(3.43)\quad I_{\nu\mu} = \int_0^1 y\, Z^2_{\nu\mu}(y)\, dy = \int_0^1 y\, J_\nu^2(\bar{\eta}_{\nu\mu} y)\, dy$$

It is seen that the integration[6] leads to the expression

$$(3.44)\quad I_{\nu\mu} = \frac{y^2}{2}\left[J_\nu^2(\bar{\eta}_{\nu\mu} y) - J_{\nu-1}(\bar{\eta}_{\nu\mu} y)\, J_{\nu+1}(\bar{\eta}_{\nu\mu} y)\right]_0^1$$

Also[7]

$$(3.45)\quad Z_{p-1}(y) = \frac{dZ_p(y)}{dy} + \frac{p}{y} Z_p(y)$$

$$(3.46)\quad -Z_{p+1}(y) = \frac{dZ_p(y)}{dy} - \frac{p}{y} Z_p(y)$$

The substitution of Equations (3.45) and (3.46) into Equation (3.44) gives the new relation

$$(3.47)\quad I_{\nu\mu} = \frac{y^2}{2}\left[J_\nu^2(\bar{\eta}_{\nu\mu} y) + J_\nu'(\bar{\eta}_{\nu\mu} y)\right]_0^1$$

where $J_\nu'(\bar{\eta}_{\nu\mu} y)$ denotes the first derivative of J_ν with respect to the argument.

Since the arguments are chosen such that $J_\nu'(\eta_{\nu\mu} y) = 0$ and the integration limits are y = 0 and y = 1, the value of $I_{\nu\mu}$ is

$$(3.48)\quad I_{\nu\mu} = J_\nu^2(\bar{\eta}_{\nu\mu} y)$$

The term $K_{\nu\mu}$ is written

$$(3.49)\quad K_{\nu\mu} = \int_0^1 y^{\nu+1} Z_{\nu\mu}(y)\, dy = \int_0^1 y^{\nu+1} J_\nu(\bar{\eta}_{\nu\mu} y)\, dy$$

Substituting the following differential equation

$$(3.50)\quad \frac{d}{dy}\left[y\, J_\nu'(\bar{\eta}_{\nu\mu} y)\right] + \left[y\, \bar{\eta}^2_{\nu\mu} - \frac{\nu^2}{y}\right] J_\nu(\bar{\eta}_{\nu\mu} y) = 0$$

into Equation (3.49), one obtains

$$(3.51)\quad K_{\nu\mu} = \frac{1}{\bar{\eta}^2_{\nu\mu}}\left\{-\int_0^1 y^\nu \frac{d}{dy}\left[y\, J_\nu'(\bar{\eta}_{\nu\mu} y)\right] dy + \nu^2 \int_0^1 y^{\nu-1} J_\nu(\bar{\eta}_{\nu\mu} y)\, dy\right.$$

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6 Ibid., p. 146

7 Ibid., p. 145

Integrating by parts, one can rewrite Equation (3.51)

$$(3.52)\quad K_{\nu\mu} = \frac{1}{\bar{\eta}_{\nu\mu}^2}\left[-y^\nu y J_\nu'(\bar{\eta}_{\nu\mu} y)\Big|_0^1 + \int_0^1 \nu y^\nu J_\nu'(\bar{\eta}_{\nu\mu}) dy + \nu^2 \int_0^1 y^{\nu-1} J_\nu(\bar{\eta}_{\nu\mu} y) dy\right]$$

Further integration and the substitution of the limits of $y = 0$ and $y = 1$, leads to the relation

$$(3.53)\quad K_{\nu\mu} = \frac{1}{\bar{\eta}_{\nu\mu}^2}\left[-J_\nu'(\bar{\eta}_{\nu\mu} y) + \nu J_\nu(\bar{\eta}_{\nu\mu} y)\right]$$

Since $J_\nu'(\eta_{\nu\mu} y) = 0$ the analytical representation of $K_{\nu\mu}$ is

$$(3.54)\quad K_{\nu\mu} = \frac{\nu}{\bar{\eta}_{\nu\mu}^2} J_\nu(\bar{\eta}_{\nu\mu} y)$$

APPENDIX F

Solution of Equation (1.33) if φ_{B_x} is Given by $x^\star$

The solution of the differential equation (1.33) is determined for a more general case. Let the coefficients of the differential equation be such that it is a power t of the longitudinal position.

$$(3.55)\quad \varphi_{B_x} = x^\star$$

This, when inserted into the differential equation gives

$$(3.56)\quad a_{o\mu}^{*\,\prime\prime} + \frac{\star}{x} a_{o\mu}^{*\,\prime} + \frac{1}{x^\star}\left(\frac{\bar{\eta}_{o\mu}^2}{\gamma+1}\right) a_{o\mu}^* = 0$$

In order to remove the inconvenient power $x^\star$ in the coefficient of $a_{o\mu}^*$, let us introduce a new independent variable u by the equation $u = x^k$ where k is a suitable constant which is chosen later. Then one obtains

$$(3.57)\quad \frac{d a_{o\mu}^*}{dx} = k\, x^{k-1} \frac{d a_{o\mu}^*}{du}$$

$$(3.58)\quad \frac{d^2 a_{o\mu}^*}{dx^2} = k^2 x^{2(k-1)} \frac{d^2 a_{o\mu}^*}{du^2} + k(k-1) x^{k-2} \frac{d a_{o\mu}^*}{du}$$

If one writes $\lambda = \frac{\bar{\eta}_{o\mu}^2}{\gamma+1}$ and inserts Equations (3.57) and (3.58) into Equation (3.56), the following equation will result

$$(3.59)\quad \frac{d^2 a_{o\mu}^*}{du^2} + \Upsilon \frac{1}{u} \frac{d a_{o\mu}^*}{du} + \frac{\lambda a_{o\mu}^*}{k^2 u^{(2+\frac{\star-2}{k})}}$$

where

(3.60) $\Upsilon = \frac{k+\chi-1}{k}$

If $k = 1 - \frac{\chi}{2}$, Equation (3.59) simplifies to

(3.61) $\frac{d^2 a^*_{o\mu}}{du^2} + \frac{\Upsilon}{u} \frac{d a^*_{o\mu}}{du} + \frac{\lambda}{k^2} a^*_{o\mu} = 0$

Equation (3.61) has the following form

(3.62) $a'' + \left(\frac{1-2s}{u}\right) a' + \left(\beta^2 + \frac{s^2-p^2}{u^2}\right) a = 0$

the solution[8] of which is given by

(3.63) $a = u^s Z_p(\beta u)$

where Z_p is a Bessel function of p order.

Let us set the following relations:

(3.64) $s = \frac{1-\Upsilon}{2}$

(3.65) $p = s = \frac{1-\Upsilon}{2}$

(3.66) $\beta = \left(\frac{\lambda}{k^2}\right)^{\frac{1}{2}}$

The solution of the differential equation (3.61) involving coefficients that are powers of x is expressed as

(3.67) $a^*_{o\mu} = u^s Z_s\left[\left(\frac{\lambda}{k^2}\right)^{\frac{1}{2}} u\right]$

where

(3.68) $s = \frac{1-\chi}{2-\chi}$

[8] Ibid., p. 146

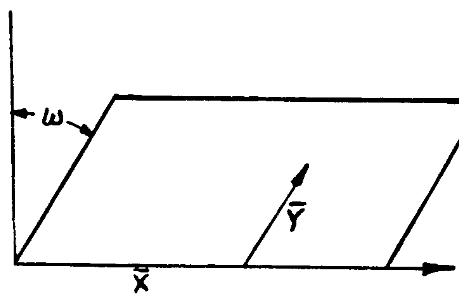

FIG. 1— CYLINDRICAL COORDINATE SYSTEM.

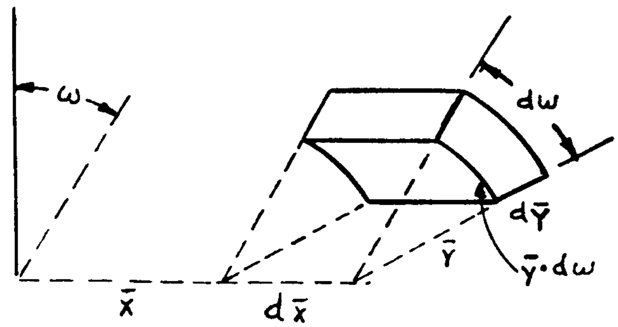

FIG. 2— SPACE ELEMENT IN CYLINDRICAL COORDINATE SYSTEM.

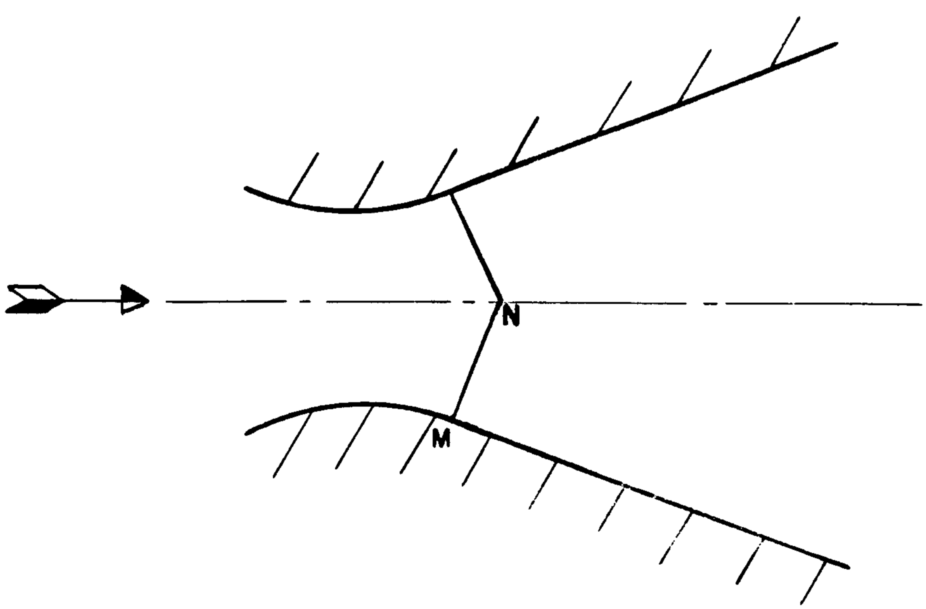

FIG. 3— PROPAGATION OF DISTURBANCES FROM SUPERSONIC JUNCTION.

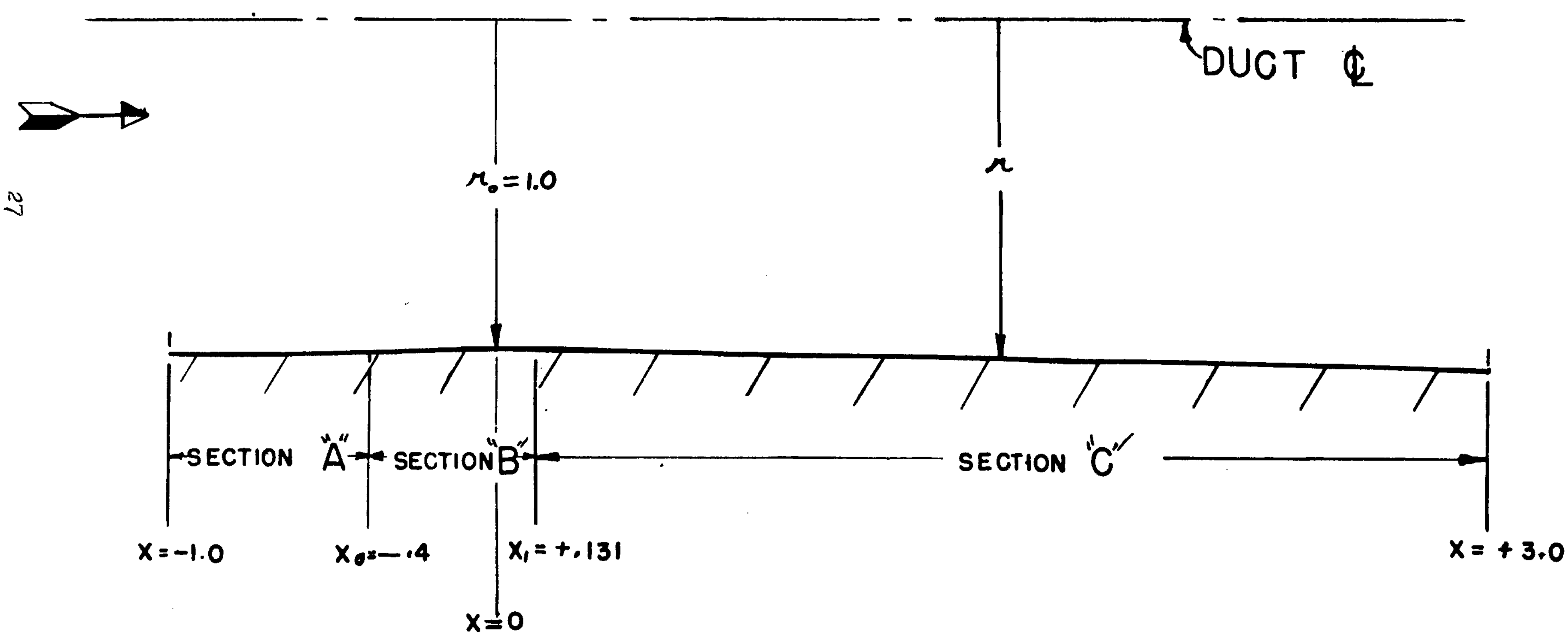

FIG.4— AXIALLY— SYMMETRIC DUCT CONTOUR

SCALE:— 1" = 2.5'

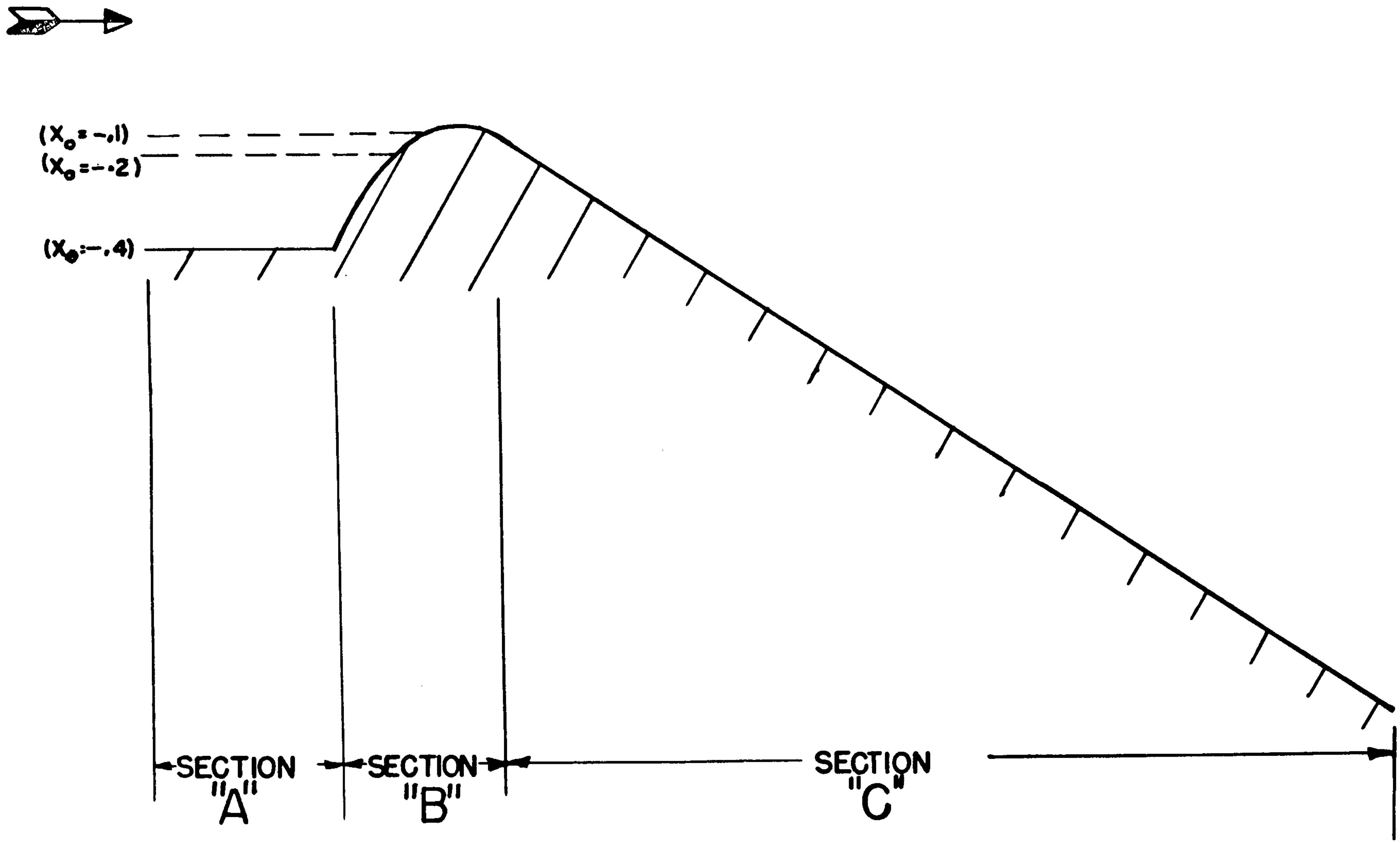

FIG.5— AXIALLY— SYMMETRIC-DUCT - CONTOUR

SCALE:— 1" = 2.5' HORIZ.
1" = .01' VERT.

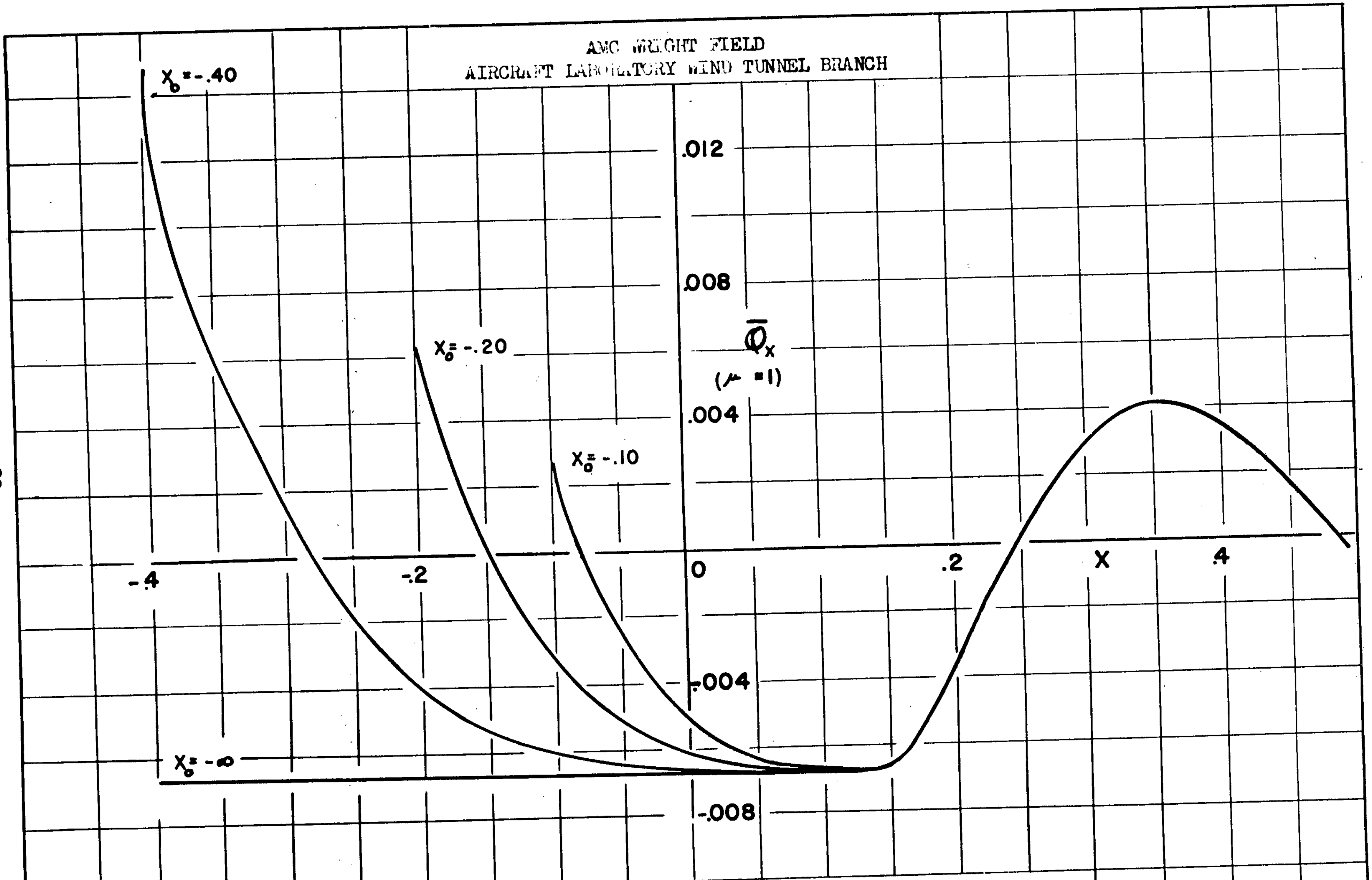

Figure 6. - Mach Number Deviation (From One-Dimensional Theory) for Different Subsonic Ducts.

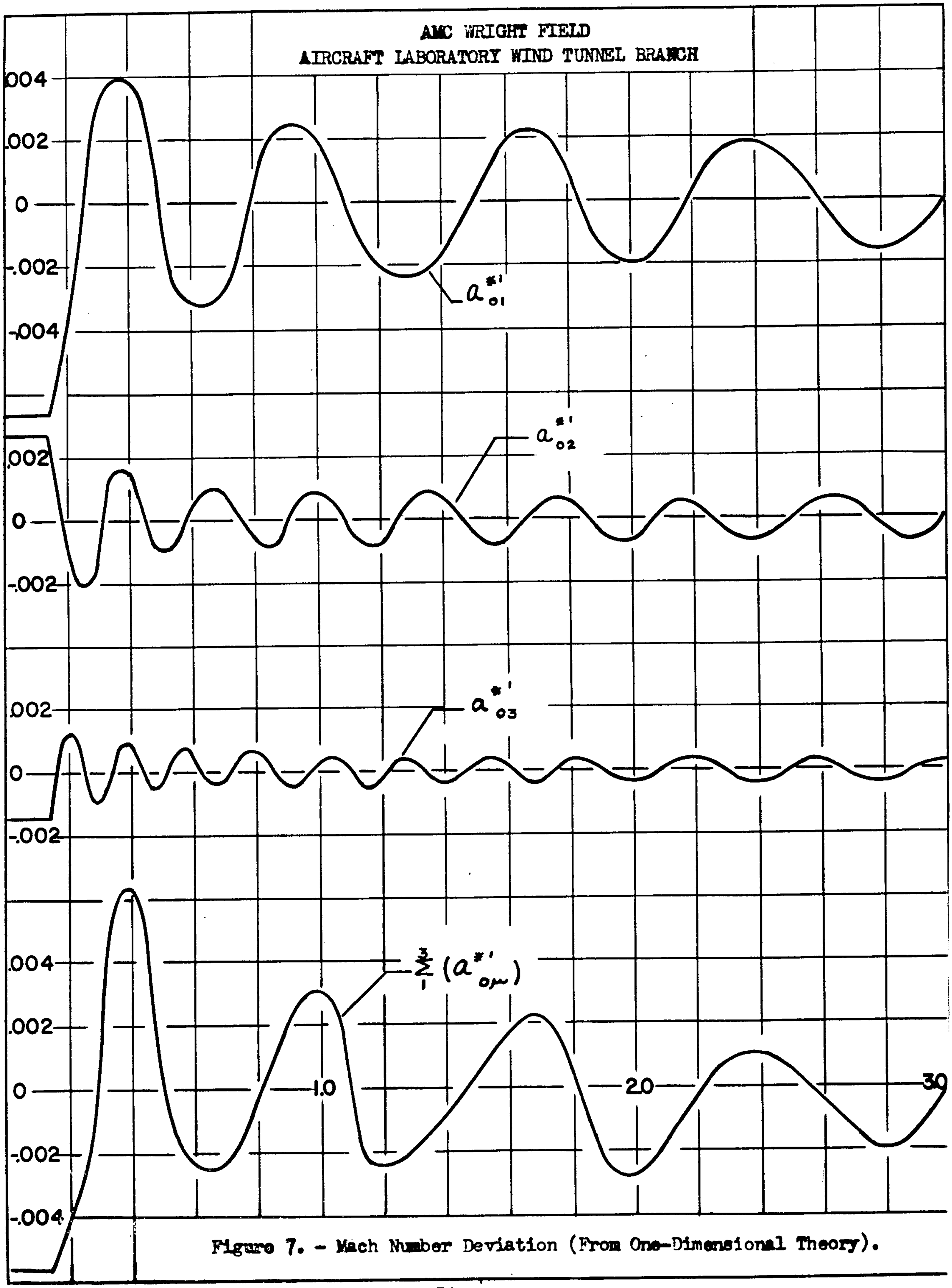

Figure 7. - Mach Number Deviation (From One-Dimensional Theory).

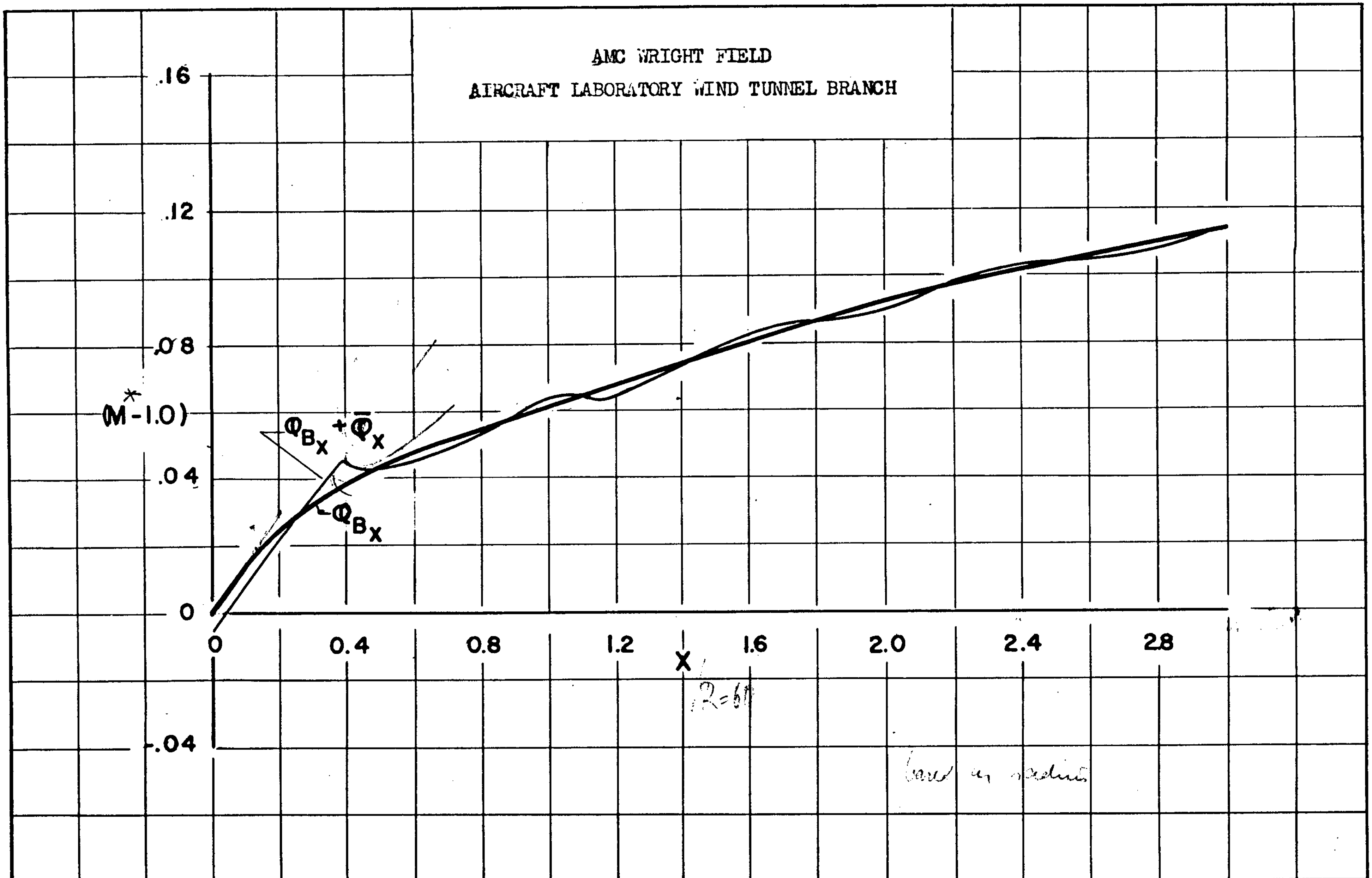

Figure 8. - Additional Mach Number Along the X-Axis.